The Propagator for A Damped Harmonic Oscillator Coupled to An Electric Field

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Abstract

In this paper, the propagator for a damped harmonic oscillator coupled to an electric field is derived by using Feynman path integral as well as Functional methods. The Lagrangian the damped harmonic oscillator coupled to electric field at most quadratic in velocity and position variables. We also found that the propagator obtained is consistent with the propagators for the simple harmonic oscillator when an electric field and a resistance constant are zero as well as $\Omega = \omega$.

Keywords: Feynman path integral, propagator, electric field, Lagrangian, damped harmonic oscillator

Introduction

In 1948, Feynman (Feynman 1948) published the article on path integral, which is a powerful and elegant method of calculation for quantum mechanical problems. The Feynman path integral expression of the propagator can be written as the summation of over all possible paths between the initial and the final points. Afterward, Feynman and Hibbs (Feynman & Hibbs 1965) published the one of famous textbook on tittle "Quantum Mechanics and Path Integrals" which deals with the path integral in theoretical and application, it has also been republished many times. Furthermore the path integral is applied to statistical physics, condense matter and quantum field theory as well. Such the path-integral method has hitherto been extremely successful in tackling nonrelativistic quantum-mechanical problems, such as the calculation of eigenvalue of harmonic oscillator (Cohen 1998, Holstein 1998), the evaluation of propagator for electron in a magnetic field and for spinless particle in a one-dimensional (Nevels, Wu & Huang 1993) as well as infinite square well (Goodman 1981), the application to spinless particle moving through potential barriers problem (Barut & Duru 1988), and so on (Inomata, Kuratsuji & Gerry 1992, Schulman 1981, Khandekar, Lawande & Bhagwat 1995, Grosche & Steiner 1997, Kleinert 1995, Amit 1984, Fried 1972, Rivers 1987, Ashok 1993).

Recently there have been a number of papers dealing with application of the path-integral method for the propagator of a damped harmonic oscillator. For example, Gerry (Gerry 1984) wished to consider a path integral quantization for a damped harmonic oscillator where he also employ a canonical transformation to reduce the problem to a harmonic oscillator, Junker and Inomata (Inomata, Kuratsuji & Gerry 1992) showed that the propagator for a time-dependent damped harmonic oscillator, and Cheng (Cheng 1984) evaluated the propagator of a damped harmonic oscillator by Montroll's method. The beyond Feynman's path integral, Urrutia and Hernándes (Urrutia & Hernándes 1984) calculated the propagator a damped harmonic oscillator by using Schwinger action principle.

The purpose of present work is to apply the path-integral method for obtaining a formal expression of the propagator for a damped harmonic oscillator coupled to an electric field. We also employ the functional methods for deriving the propagator for this system described by a Lagrangian at most quadratic in the velocity and position variable.

In their well-known exposition of the path integral approach to the propagator, Feynman and Hibbs (Feynman & Hibbs 1965) show that the propagator for a particle mass m can be written as

$$K(\vec{r},T;\vec{r}_0,0) = N\int D\vec{r}\, e^{iS/\hbar} \tag{1}$$

with $D\vec{r} \equiv \lim_{N\to\infty}\prod_{l=1}^{N-1} d^3 r_l$, a normalization factor given by $N \equiv \lim_{N\to\infty}\prod_{k=1}^{N-1}\left(m/2\pi i\hbar\varepsilon\right)$, and where the time interval T has been divided into N equal pieces of length ε. The action S is defined as the integral of the Lagrangian function over the period of time T:

$$S = \int_0^T L(\dot{\vec{r}},\vec{r})\,dt \tag{2}$$

The propagator in Eq. (1) arises as a sum over all paths beginning at the space-time coordinate $(\vec{r}_0,0)$ and ending at the space-time coordinate $(\vec{r},T)$, and conveys the particle from an initial state $\psi(\vec{r},T)$, namely.

$$\psi\left(\vec{r},T\right) = \int d^3 r_0 K(\vec{r},T;\vec{r}_0,0)\psi\left(\vec{r}_0,0\right) \tag{3}$$

The Lagrangian of the damped harmonic oscillator coupled to electric field can be written as $(i = 1,2,3)$

$$L = e^{(b/m)t}\left(\frac{1}{2}m\dot{x}_i^2 - \frac{1}{2}m\omega^2 x_i^2\right) + qE_i x_i \tag{4}$$

where m is the mass of the particle, b is the resistance constant, ω is natural frequency of oscillation, q is an electric charged, and E_i is an electric field. We will assume that the motion takes place between fixed initial and final points and calculate the propagator for the system. We only study the motion for under-damped case ($b/m < 2\omega$).

The organization of the rest of the paper is as follows: in Sec. II we will derive the propagator for a damped harmonic oscillator coupled to an electric field is derived by employing Functional methods (see also in the work of Poon and Muñoz (Poon & Muñoz 1998)). In Sec. III we conclude with a summary of our results.

II. DERIVATION OF THE PROPAGATOR

Since Eq.(1) involves an integral over all possible paths, $\vec{x}(t)$ may be written as a linear combination of the classical path $\vec{x}(t)$, the solution of the Euler–Lagrange equations obtained from L, and the deviated path $\vec{y}(t)$, the quantum fluctuation:

$$r_i(t) = x_i(t) + y_i(t) \tag{5}$$

with the boundary conditions,

$$x_i(T) = x_b, \quad x_i(0) = x_a \tag{6}$$

$$y_i(T) = y_i(0) = 0 \tag{7}$$

Covering all paths is now accomplished by integrating over the fluctuation $\vec{y}(t)$. The classical solution $x_i(t)$ can be found from the Euler–Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k} = 0 \quad (k = 1,2,3) \tag{8}$$

The equation of motion is

$$(m\ddot{x}_k + b\dot{x}_k + m\omega^2 x_k)e^{(b/m)t} - qE_k = 0 \tag{9}$$

We now separate the classical contribution to the action, $S = S_c + \bar{S}$, with

$$S_c = \int_0^T \left(e^{(b/m)t}\left(\frac{1}{2}m\dot{x}_i^2 - \frac{1}{2}m\omega^2 x_i^2\right) + qE_i x_i \right) dt, \tag{10}$$

$$= \frac{1}{2}m x_i \dot{x}_i e^{(b/m)t}\Big|_0^T - \frac{1}{2}\int_0^T \left(m\ddot{x}_i + b\dot{x}_i + m\omega^2 x_i\right) x_i e^{(b/m)t}\, dt + \int_0^T qE_i x_i dt \tag{11}$$

From the equation of motion (9) leads to

$$S_c = \frac{1}{2}m x_i \dot{x}_i e^{(b/m)t}\Big|_0^T + \frac{1}{2}\int_0^T qE_i x_i\, dt, \tag{12}$$

Solving the equations of motion with the boundary conditions (6), the classical action is determined to be

$$S_c = \frac{m\Omega}{2\sin\Omega T}\Big[\left(x_a^2 + x_b^2 e^{(bT/m)}\right)\cos\Omega T - 2x_a x_b e^{(bT/2m)} + \left(\frac{b}{2m\Omega}\right)\left(x_a^2 - x_b^2 e^{(bT/m)}\right)\sin\Omega T\Big] + \frac{qE_i\Omega}{2\omega^2\sin\Omega T}\Big\{\left(e^{(bT/2m)} - \cos\Omega T - b\sin\Omega T\right)x_a + \left(e^{(bT/2m)} - e^{(bT/m)}\cos\Omega T + \frac{be^{(bT/m)}}{2m\Omega}\sin\Omega T\right)x_b + \frac{1}{b^2+4m^2\Omega^2}\Big[qE_i\Big[\frac{T}{m\Omega}\left(b^2+4m^2\Omega^2\right)\sin\Omega T + 8m\cos\Omega T - 4me^{(-bT/2m)}\left(1+e^{(bT/m)}\right)\Big] + 2m\omega^2\left[b\left(x_a - x_b\right)\sin\Omega T - 2m\left(x_a + x_b\right)\cos\Omega T\right] + 4m^2\omega^2\left(x_a e^{(-bT/2m)} + x_b e^{(bT/2m)}\right)\Big]\Big\} \tag{13}$$

where $\Omega = \left(\omega^2 - (b/2m)^2\right)$, and for the contribution action

$$\bar{S} = \int_0^T\left(m\dot{x}_i\dot{y}_i + \frac{1}{2}m\dot{y}_j^2 - \frac{1}{2}m\omega^2(x_i y_j + x_j y_i) - \frac{1}{2}m\omega^2 y_i y_j\right)e^{(b/m)t} + \left(qE_i y_i\right)dt, \tag{14}$$

Applying the condition (7) after integration by parts,

$$\bar{S} = \int_0^T dt\Big[\left(-m\ddot{x}_i y_i - b\dot{x}_i y_i - m\omega^2 x_i y_i\right)e^{(b/m)t} + qE_i y_i + \left(\frac{1}{2}m\dot{y}_i^2 + \frac{1}{2}m\omega^2 y_i y_j\right)e^{(b/m)t}\Big] \tag{15}$$

The first four terms,

$$\begin{aligned}&\left(-m\ddot{x}_i y_i - b\dot{x}_i y_j - m\omega^2 x_i y_i\right)e^{(b/m)t} - qE_i y_i \\ &= \left[\left(-m\ddot{x}_k - b\dot{x}_k - m\omega^2 x_k\right)e^{(b/m)t} - qE_k\right]y_k\end{aligned} \tag{16}$$

vanish because of the equation of motion (9). Now

$$\begin{aligned}\bar{S} &= \int_0^T\left(\frac{1}{2}m\dot{y}_i^2 + \frac{1}{2}m\omega^2 y_i y_j\right)e^{(b/m)t}dt \\ &= \int_0^T y_i\left(-\frac{1}{2}m\delta_{ij}\ddot{y}_i - \frac{b}{2}\delta_{ij}\dot{y}_i - \frac{1}{2}m\omega^2\right)y_j e^{(b/m)t}dt \\ &= \int_0^T y_i \Lambda y_j e^{(b/m)t}dt\end{aligned} \tag{17}$$

where

$$\Lambda_{ij} = -\frac{1}{2}m\delta_{ij}\ddot{y}_i - \frac{b}{2}\delta_{ij}\dot{y}_i - \frac{1}{2}m\omega^2 \tag{18}$$

It will be easier to manipulate (17) in matrix notation:

$$\bar{S} = \int_0^T y_i^T \Lambda y_j e^{(b/m)t}dt \tag{19}$$

A convenient way to integrate over the quantum fluctuations $\vec{y}(t)$ is to first expand these in terms of the eigenfunctions of Λ. Since this is a 3×3 matrix differential operator, eigenfunctions satisfying the boundary conditions (7) should be associated with (a triple infinity of) discrete eigenvalues. We shall use $(n_1,n_2,n_3)\equiv n_\alpha$ to label these eigenvalues and eigenfunctions, i.e.,

$$\Lambda u_{n_\alpha} = \lambda_{n_\alpha} u_{n_\alpha} \tag{20a}$$

$$u_{n_\alpha}(0) = u_{n_\alpha}(T) = 0 \tag{20b}$$

defines the eigenvalue–eigenfunction problem for Λ. With this notation, the expansion we seek reads

$$y = \sum_{n_\alpha} c_{n_\alpha} u_{n_\alpha} \tag{21}$$

Then

$$\Lambda y = \sum_{n_\alpha} c_{n_\alpha} \lambda_{n_\alpha} u_{n_\alpha} \tag{22}$$

and Eq. (19) becomes

$$\bar{S} = \int_0^T \sum_{m_\beta, n_\alpha} c_{m_\beta} c_{n_\alpha} u_{m_\beta}^T \lambda_{n_\alpha} u_{n_\alpha} e^{(b/m)t} dt \tag{23}$$

The orthornormality of the eigenfunctions, $\int_0^T u_{m_\beta}^T u_{n_\alpha} e^{(b/m)t} dt = e^{(b/m)T} \delta_{n_\beta n\alpha}$ yields

$$\bar{S} = \sum_{m_\beta, n_\alpha} c_{n_\alpha}^2 \lambda_{n_\alpha} e^{(b/m)T} \tag{24}$$

leads to the propagator

$$K(\vec{r},T;\vec{r}_0,0) = \mathcal{N} \exp\left(iS_c/\hbar\right) \int \exp\left(i\bar{S}/\hbar\right) D\vec{y} \tag{25}$$

where the remaining path integral has the explicit form

$$\int \exp\left(i\bar{S}/\hbar\right) D\vec{y} = \int \lim_{N\to\infty} \prod_{l=1}^{N-1} d\vec{y} \exp\left(i \sum_{n_\alpha} \lambda_{n_\alpha} c_{n_\alpha}^2 e^{(b/m)T} / \hbar \right) \tag{26}$$

Since the expression (21) for $\vec{y}$ is linear in c_{n_α}, the Jacobian J of the transformation from $\vec{y}$ to c_{n_α},

$$\prod_l d\vec{y} \to J \prod_{n_\alpha} c_{n_\alpha} \tag{27}$$

does not depend on the c_{n_α}, so that

$$\int e^{i\bar{S}/\hbar} D\vec{y} = J \prod_{n_\alpha} \int_{-\infty}^{\infty} e^{i\lambda_{n_\alpha} c_{n_k}^2 e^{(b/m)T}/\hbar} dc_{n_\alpha} = J \prod_{n_\alpha} \sqrt{\frac{i\pi\hbar}{\lambda_{n_\alpha} e^{(b/m)T}}} \tag{28}$$

or

$$\int e^{i\bar{S}/\hbar} D\bar{y} = J(\det\Lambda)^{-1/2} e^{(-b/2m)T} \prod_{n_\alpha} \sqrt{i\pi\hbar} \quad (29)$$

where $\det\Lambda \equiv \prod_{n_\alpha} \lambda_{n_\alpha}$. Hence

$$K(\vec{r},T;\vec{r}_0,0) = NJe^{iS_c/\hbar}(\det\Lambda)^{-1/2} e^{(-b/2m)T} \prod_{n_\alpha} \sqrt{i\pi\hbar} \quad (30)$$

We may avoid a direct calculation of N , J , and the infinite product in Eq.(30) by appealing to the free-particle propagator $K^{(F)}$. Therefore,

$$K^{(F)}(\vec{r},T;\vec{r}_0,0) = NJe^{iS_c^{(F)}/\hbar}(\det\Lambda^{(F)})^{-1/2} \prod_{n_\alpha} \sqrt{i\pi\hbar} \quad (31)$$

We may then write Eq.(30) in the form

$$K(\vec{r},T;\vec{r}_0,0) = K^{(F)}(\vec{r},T;\vec{r}_0,0)\left(\frac{\det\Lambda^{(F)}}{\det\Lambda}\right)^{-1/2} e^{i\left(S_c - S_c^{(F)}\right)/\hbar} \quad (32)$$

The free particle propagator may be found in numerous textbooks and journal articles (where it is usually computed directly from the Schrödinger equation or by means of operator methods):

$$K^{(F)}(\vec{r},T;\vec{r}_0,0) = \left(\frac{m}{2\pi i\hbar T}\right)^{(3/2)} e^{iS_c^{(F)}/\hbar} \quad (33)$$

Thus our final result is

$$K(\vec{r},T;\vec{r}_0,0) = \left(\frac{me^{(b/m)T}}{2\pi i\hbar T}\right)^{3/2} \left(\frac{\det\Lambda^{(F)}}{\det\Lambda}\right)^{1/2} e^{iS_c/\hbar}$$

(34)

For the applications that follow it will be convenient to have an explicit expression for $\det\Lambda^{(F)}$. For the free particle, $\Lambda_{ij}^{(F)} = -(m/2)\delta_{ij}(d^2/dt^2)$. The eigenfunctions satisfying Eq. (20a) are $u_{n_\alpha}(t) = \beta\cos\left(\sqrt{(2\lambda_{n_\alpha}^{(F)}/m)t}\right) + \gamma\sin\left(\sqrt{(2\lambda_{n_\alpha}^{(F)}/m)t}\right)$ with β and γ arbitrary column vectors. Imposing the boundary conditions (20b), we find the three eigenvalues: $\lambda_{n_\alpha}^{(F)} = (m/2)(n_\alpha\pi/T)^2$ where $n_\alpha = 1,2,3,...$ for each $\alpha = 1,2,3,...$. Consequently,

$$\det\Lambda^{(F)} \equiv \prod_{n_\alpha} \lambda_{n_\alpha}^{(F)}$$

$$= \prod_{n_1} \frac{m}{2}\left(\frac{n_1\pi}{T}\right)^2 \prod_{n_2} \frac{m}{2}\left(\frac{n_2\pi}{T}\right)^2 \prod_{n_3} \frac{m}{2}\left(\frac{n_3\pi}{T}\right)^2 \quad (35)$$

or

$$\left(\det\Lambda^{(F)}\right)^{-1/2} = \prod_{n_1,n_2,n_3} \left(\frac{2T}{m\pi^2}\right)^{3/2} (n_1 n_2 n_3)^{-1} \quad (36)$$

In our problem, we will find $\det\Lambda$, we need to solve for the eigenvalues of Λ from (20a):

$$\frac{m}{2}\left(\frac{d^2}{dt^2}+\frac{b}{m}\frac{d}{dt}+\omega^2\right)u_{n_\alpha}=-\lambda_{n_\alpha}u_{n_\alpha} \tag{37}$$

or

$$\ddot{u}_{n_\alpha}+\frac{b}{m}\dot{u}_{n_\alpha}+\left(\omega^2+\frac{2}{m}\lambda_{n_\alpha}\right)u_{n_\alpha}=0 \tag{38}$$

Solving Eq. (38) with the boundary conditions (20b), we obtain the eigenvalues

$$\lambda_{n_\alpha}=\frac{1}{2}m\left[\left(\frac{\pi}{T}\right)^2 n_{n_\alpha}^2-\Omega^2\right], \quad (n_\alpha=1,2,3,\ldots) \tag{39}$$

Hence

$$(\det\Lambda)^{-1/2}=\left(\frac{\Omega T}{\sin\Omega T}\right)\prod_{n_1,n_2,n_3}\left(\frac{2}{m}\right)^{3/2}\left(\frac{T}{\pi}\right)^2(n_1n_2n_3)^{-1} \tag{40}$$

where we have used the formula $\sin x = x\prod_{n=1}^{\infty}(1-x^2/n^2\pi^2)$ (Arfken & Weber 1995) .

Comparing with Eq. (36), we find

$$\left(\frac{\det\Lambda^{(F)}}{\det\Lambda}\right)^{1/2}=\left(\frac{\Omega T}{\sin\Omega T}\right)^{3/2} \tag{41}$$

Now the propagator (34) can be written as

$$K(\vec{r},T;\vec{r}_0,0)=\left(\frac{m\Omega e^{(b/m)T}}{2\pi i\hbar\sin\Omega T}\right)^{3/2}e^{iS_c/\hbar} \tag{42}$$

with S_c given by Eq.(13). This solution is consistent with the propagators for the simple harmonic oscillator ($E_i\to 0$, $\Omega=\omega$ and $b\to 0$).

III. CONCLUSIONS

We have presented a strictly analytical method by which the full propagator for the under-damped harmonic oscillator coupled to an electric field be obtained using Feynman's path integral approach. The functional methods proposed in this paper has been generalized by Poon and Muñoz (Poon & Muñoz 1999) for the Lagrangian at most quadratic in the velocity and position variables. However, the propagator in (42) of the under-damped harmonic oscillator coupled to an electric field is new. This solution is consistent with the propagators for the simple harmonic oscillator when an electric field and a resistance constant are zero as well as $\Omega=\omega$.

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